Chapter 2 : Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Lecture 10 : Exercises & Correction

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Exercice 6

A.

Prove the second De Morgan law in Table 1 by showing that if A and B are sets, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$

a) by showing each side is a subset of the other side.

b) using a membership table.

B.

Let A and B be sets. Show that

a) $(A \cap B) \subseteq A$.

b) $A \subseteq (A \cup B)$.

c) $A - B \subseteq A$.

d) $A \cap (B - A) = \emptyset$.

e) $A \cup (B - A) = A \cup B$.

Solution Exercise 6-A

(a) To proof: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

PROOF

FIRST PART Let $x \in \overline{A \cup B}$.

Using the definition of the complement, x is in the complement of $A \cup B$ when x is not in $A \cup B$

$$x \notin A \cup B$$

If x is not an element of the union, then x is not an element of either set:

$$x \notin A \wedge x \notin B$$

Using the definition of the complement, x is in the complement of the set when x is not in the set:

$$x \in \overline{A} \wedge x \in \overline{B}$$

Using the definition of the intersection, x is in the intersection of two sets when it is both sets.

$$x \in \overline{A} \cap \overline{B}$$

We have then shown $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

SECOND PART Let $x \in \overline{A} \cap \overline{B}$.

Using the definition of the intersection, x is in the intersection of two sets when it is both sets.

$$x \in \overline{A} \wedge x \in \overline{B}$$

Using the definition of the complement, x is in the complement of the set when x is not in the set:

$$x \notin A \wedge x \notin B$$

Or equivalently:

$$\neg(x \in A) \wedge \neg(x \in B)$$

Use De Morgan's law for propositions:

$$\neg(x \in A \vee x \in B)$$

Using the definition of the union, x is the union of the sets if x is in one of the sets (or both).

$$\neg(x \in A \cup B)$$

Or equivalently:

$$x \notin A \cup B$$

Using the definition of the complement, x is in the complement of the set when x is not in the set:

$$x \in \overline{A \cup B}$$

We have then shown $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

CONCLUSION We obtained $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$, thus the two sets have to be equal.

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

(b) If x is an element, then 1 represents that the element is in the set and 0 represents that the element is not in the set.

A	B	$A \cup B$	$\overline{A}$	$\overline{B}$	$\overline{\mathbf{A} \cup \mathbf{B}}$	$\overline{\mathbf{A}} \cap \overline{\mathbf{B}}$
0	0	0	1	1	**1**	**1**
0	1	1	1	0	**0**	**0**
1	0	1	0	1	**0**	**0**
1	1	1	0	0	**0**	**0**

Since the last two columns have the same values, the two expressions are equal.

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Solution Exercise 6-B

(a) To proof: $(A \cap B) \subseteq A$

PROOF

Let $x \in A \cap B$.

Using the definition of the intersection, x is in the intersection when it is in both sets:

$$x \in A \wedge x \in B$$

Using simplification of propositions, we then know that x has to be an element of A:

$$x \in A$$

By the definition of a subset, we have then shown $(A \cap B) \subseteq A$.
$\square$

(b) To proof: $A \subseteq (A \cup B)$

PROOF

Let $x \in A$.

Using addition of propositions:

$$x \in A \vee x \in B$$

Using the definition of the union, an element is in the union, when it is in one of the sets (or both)

$$x \in A \cup B$$

By the definition of a subset, we have then shown $A \subseteq (A \cup B)$.

(c) To proof: $A - B \subseteq A$

PROOF

Let $x \in A - B$.

Using the definition of the difference, we then know that x is in A and x is not in B.

$$x \in A \wedge x \notin B$$

Using simplification of propositions, we then know that x has to be an element of A:

$$x \in A$$

By the definition of a subset, we have then shown $A - B \subseteq A$.

(d) To proof: $A \cap (B - A) = \emptyset$

PROOF

FIRST PART Let $x \in A \cap (B - A)$.

Using the definition of the intersection, x is in the intersection when it is in both sets:

$$x \in A \land x \in B - A$$

Using the definition of the difference $B - A$, we then know that x is in B and x is not in A.

$$x \in A \land (x \in B \land \neg(x \in A))$$

Use commutative law for propositions:

$$x \in A \land (\neg(x \in A) \land x \in B)$$

Use associative law for propositions:

$$(x \in A \land \neg(x \in A)) \land x \in B$$

Use negation law for propositions:

$$\mathbf{F} \land x \in B$$

Use domination law for propositions:

$$\mathbf{F}$$

The emptyset does not contain any elements, thus the statement $x \in \emptyset$ is always false.

$$x \in \emptyset$$

By the definition of a subset, we have then shown $A \cap (B - A) \subseteq \emptyset$.

SECOND PART

The empty set is a subset of every set:

$$\emptyset \subseteq A \cap (B - A)$$

CONCLUSION

Since $A \cap (B - A) \subseteq \emptyset$ and $\emptyset \subseteq A \cap (B - A)$, the two sets have to be the same:

$$A \cap (B - A) = \emptyset$$

(e) To proof: $A \cup (B - A) = A \cup B$

PROOF

FIRST PART Let $x \in A \cup (B - A)$.

Using the definition of the union, x is in the union when it is in one of the sets:

$$x \in A \lor x \in B - A$$

Using the definition of the difference $B - A$, we then know that x is in B and x is not in A.

$$x \in A \lor (x \in B \land \neg(x \in A))$$

Use commutative law for propositions:

$$x \in A \lor (\neg(x \in A) \land x \in B)$$

Use simplifcation for propositions:

$$x \in A \lor (x \in B)$$

Using the definition of the union, x is in the union when it is in one of the sets:

$$x \in A \cup B$$

By the definition of a subset, we have then shown $A \cap (B - A) \subseteq A \cup B$.

SECOND PART Let $x \in A \cup B$

The empty set is a subset of every set:

$$\emptyset \subseteq A \cap (B - A)$$

Using the definition of the union, x is in the union when it is in one of the sets:

$$x \in A \vee (x \in B)$$

Use addition for propositions:

$$x \in A \vee (\neg(x \in A) \wedge x \in B)$$

Use commutative law for propositions:

$$x \in A \vee (x \in B \wedge \neg(x \in A))$$

Using the definition of the difference $B - A$, we then know that x is in B and x is not in A.

$$x \in A \vee x \in B - A$$

Using the definition of the union, x is in the union when it is in one of the sets:

$$x \in A \cup (B - A)$$

By the definition of a subset, we have then shown $A \cup B \subseteq A \cup (B - A)$.

CONCLUSION

Since $A \cap (B - A) \subseteq A \cup B$ and $A \cup B \subseteq A \cap (B - A)$, the two sets have to be the same:

$$A \cap (B - A) = A \cup B$$

Exercice 7

A.

Show that if A, B, and C are sets, then $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

a) by showing each side is a subset of the other side.
b) using a membership table.

B.

Let A, B, and C be sets. Show that

a) $(A \cup B) \subseteq (A \cup B \cup C)$.
b) $(A \cap B \cap C) \subseteq (A \cap B)$.
c) $(A - B) - C \subseteq A - C$.
d) $(A - C) \cap (C - B) = \emptyset$.
e) $(B - A) \cup (C - A) = (B \cup C) - A$.

C.

Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

a) $A \cap (B \cup C)$ **b)** $\overline{A} \cap \overline{B} \cap \overline{C}$
c) $(A - B) \cup (A - C) \cup (B - C)$

(a) To proof: $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

PROOF

FIRST PART Let $x \in \overline{A \cap B \cap C}$.

Using the definition of the complement, x is in the complement of $A \cap B \cap C$ when x is not in $A \cap B \cap C$

$$\neg(x \in A \cap B \cap C)$$

Using the definition of the intersection, x is in the intersection of two sets when it is both sets.

$$\neg(x \in A \wedge x \in B \wedge x \in C)$$

Using De Morgan's law for propositions:

$$\neg(x \in A) \vee \neg(x \in B) \vee \neg(x \in C)$$

Using the definition of the complement, x is in the complement of a set when x is not in the set.

$$x \in \overline{A} \vee x \in \overline{B} \vee x \in \overline{C}$$

Using the definition of the union, x is in the union of two sets when it is either set.

$$x \in \overline{A} \cup \overline{B} \cup \overline{C}$$

By the definition of a subset, we have then shown $\overline{A \cap B \cap C} \subseteq \overline{A} \cup \overline{B} \cup \overline{C}$.

SECOND PART Let $x \in \overline{A} \cup \overline{B} \cup \overline{C}$.

Using the definition of the union, x is the union of the sets if x is in one of the sets (or both).

$$x \in \overline{A} \vee x \in \overline{B} \vee x \in \overline{C}$$

Using the definition of the complement, x is in the complement of the set when x is not in the set:

$$\neg(x \in A) \vee \neg(x \in B) \vee \neg(x \in C)$$

Use De Morgan's law for propositions:

$$\neg(x \in A \wedge x \in B \wedge x \in C)$$

Using the definition of the intersection, x is in the intersection of two sets when it is both sets.

$$\neg(x \in A \cap B \cap C)$$

Using the definition of the complement, x is in the complement of the set when x is not in the set:

$$x \in \overline{A \cap B \cap C}$$

We have then shown $\overline{A} \cup \overline{B} \cup \overline{C} \subseteq \overline{A \cap B \cap C}$.

CONCLUSION We obtained $\overline{A \cap B \cap C} \subseteq \overline{A} \cup \overline{B} \cup \overline{C}$ and $\overline{A} \cup \overline{B} \cup \overline{C} \subseteq \overline{A \cap B \cap C}$, thus the two sets have to be equal.

$$\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$$

(b) If x is an element, then 1 represents that the element is in the set and 0 represents that the element is not in the set.

A	B	C	$A \cap B \cap C$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}}$	$\overline{\mathbf{A}} \cup \overline{\mathbf{B}} \cup \overline{\mathbf{C}}$
0	0	0	0	1	1	1	**1**	**1**
0	0	1	0	1	1	0	**1**	**1**
0	1	0	0	1	0	1	**1**	**1**
0	1	1	0	1	0	0	**1**	**1**
1	0	0	0	0	1	1	**1**	**1**
1	0	1	0	0	1	0	**1**	**1**
1	1	0	0	0	0	1	**1**	**1**
1	1	1	1	0	0	0	**0**	**0**

Since the last two columns have the same values, the two expressions are equal.

$$\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$$

a) To prove $\overline{A \cap A \cap B} = \overline{A} \cup \overline{B} \cup \overline{C}$

First we prove $\overline{A \cap A \cap B} \subset \overline{A} \cup \overline{B} \cup \overline{C}$

Let $x \in \overline{A \cap A \cap B}$

$\Rightarrow x \notin A \cap A \cap B$

$\Rightarrow x \notin A \cap B \quad \text{or} \quad x \notin C$

$\Rightarrow x \notin A \quad \text{or} \quad x \notin B \quad \text{or} \quad x \notin C$

$\Rightarrow x \in \overline{A} \quad \text{or} \quad x \in \overline{B} \quad \text{or} \quad x \in \overline{C}$

$\Rightarrow x \in \overline{A} \cup \overline{B} \cup \overline{C}$

$\Rightarrow \overline{A \cap A \cap B} \subset \overline{A} \cup \overline{B} \cup \overline{C}$

To prove equality we show $\overline{A} \cup \overline{B} \cup \overline{C} \subset \overline{A \cap A \cap B}$

Let $x \in \overline{A} \cup \overline{B} \cup \overline{C}$

$\Rightarrow x \in \overline{A} \cup \overline{B} \quad \text{or} \quad x \in \overline{C}$

$\Rightarrow x \in \overline{A} \quad \text{or} \quad x \in \overline{B} \quad \text{or} \quad x \in \overline{C}$

$\Rightarrow x \notin A \quad \text{or} \quad x \notin B \quad \text{or} \quad x \notin C$

$\Rightarrow x \notin A \cap B \quad \text{or} \quad x \notin C$

$\Rightarrow x \notin A \cap B \cap C$

$\Rightarrow x \in \overline{A \cap B \cap C}$

SOLUTION

(a) To proof: $(A \cup B) \subseteq (A \cup B \cup C)$

PROOF

Let $x \in A \cup B$.

Using the definition of the union, an element is in the union, when it is in one of the sets (or both)

$$x \in A \vee x \in B$$

Using addition of propositions:

$$x \in A \vee x \in B \vee x \in C$$

Using the definition of the union, an element is in the union, when it is in one of the sets (or both)

$$x \in A \cup B \cup C$$

By the definition of a subset, we have then shown $(A \cup B) \subseteq (A \cup B \cup C)$.
□

(b) To proof: $(A \cap B \cap C) \subseteq (A \cap B)$

PROOF

Let $x \in A \cap B \cap C$.

Using the definition of the intersection, x is in the intersection when it is in both sets:

$$x \in A \wedge x \in B \wedge x \in C$$

Use simplification of propositions:

$$x \in A \wedge x \in B$$

Using the definition of the intersection, x is in the intersection when it is in both sets:

$$x \in A \cap B$$

By the definition of a subset, we have then shown $(A \cap B \cap C) \subseteq (A \cap B)$.

(c) To proof: $(A - B) - C \subseteq A - C$

PROOF

Let $x \in (A - B) - C$.

Using the definition of the difference, we then know that x is in $A - B$ and x is not in C.

$$x \in (A - B) \land \neg(x \in C)$$

Using the definition of the difference, we then know that x is in A and x is not in B.

$$x \in A \land \neg(x \in B) \land \neg(x \in C)$$

Use simplification of propositions:

$$x \in A \land \neg(x \in C)$$

Using the definition of the difference, we then know that x has to be in the difference $A - C$.

$$x \in A - C$$

By the definition of a subset, we have then shown $(A - B) - C \subseteq A - C$.

(d) To proof: $(A - C) \cap (C - B) = \emptyset$

PROOF

FIRST PART Let $x \in (A - C) \cap (C - B)$.

Using the definition of the intersection, x is in the intersection when it is in both sets:

$$x \in (A - C) \wedge x \in (C - B)$$

Using the definition of the difference $A - C$, we then know that x is in A and x is not in C (similar for $C - B$).

$$x \in A \wedge \neg(x \in C) \wedge x \in C \wedge \neg(x \in B))$$

Use negation law for propositions:

$$x \in A \wedge \mathbf{F} \wedge \neg(x \in B))$$

Use domination law for propositions:

$$\mathbf{F}$$

The emptyset does not contain any elements, thus the statement $x \in \emptyset$ is always false.

$$x \in \emptyset$$

By the definition of a subset, we have then shown $(A - C) \cap (C - B) \subseteq \emptyset$.
SECOND PART

The empty set is a subset of every set:

$$\emptyset \subseteq (A - C)$$

CONCLUSION

Since $(A - C) \cap (C - B) \subseteq \emptyset$ and $\emptyset \subseteq (A - C)$, the two sets have to be the same:

$$(A - C) \cap (C - B) = \emptyset$$

(e) To proof: $(B - A) \cup (C - A) = (B \cup C) - A$

PROOF

FIRST PART Let $x \in (B - A) \cup (C - A)$.

Using the definition of the union, x is in the union when it is in one of the sets:

$$x \in (B - A) \vee x \in (C - A)$$

Using the definition of the difference $B - A$, we then know that x is in B and x is not in A (Similar for $C - A$).

$$(x \in B \wedge \neg(x \in A)) \vee (x \in C \wedge \neg(x \in A))$$

Use distributive law for propositions:

$$(x \in B \vee x \in C) \wedge \neg(x \in A)$$

Using the definition of the union, x is in the union when it is in one of the sets:

$$(x \in B \vee C) \wedge \neg(x \in A)$$

Using the definition of the difference, we then know that x is in $(B \cup C) - A$.

$$x \in (B \cup C) - A$$

By the definition of a subset, we have then shown $(B - A) \cup (C - A) \subseteq (B \cup C) - A$.

SECOND PART Let $x \in (B \cup C) - A$

Using the definition of the difference $(B \cup C) - A$, we then know that x is in $B \cup C$ and x is not in A (Similar for $C - A$).

$$x \in B \cup C \wedge \neg(x \in A)$$

Using the definition of the union, x is in the union when it is in one of the sets:

$$(x \in B \vee x \in C) \wedge \neg(x \in A)$$

Use distributive law for propositions:

$$(x \in B \wedge \neg(x \in A)) \vee (x \in C \wedge \neg(x \in A))$$

Using the definition of the difference, we then obtain:

$$x \in B - A \vee x \in C - A$$

Using the definition of the union, x is in the union when it is in one of the sets:

$$x \in (B - A) \cup (C - A)$$

By the definition of a subset, we have then shown $(B \cup C) - A \subseteq (B - A) \cup (C - A)$.

CONCLUSION

Since $(B - A) \cup (C - A) \subseteq (B \cup C) - A$ and $(B \cup C) - A \subseteq (B - A) \cup (C - A)$, the two sets have to be the same:

$$(B - A) \cup (C - A) = (B \cup C) - A$$

Exercice 8

A.

Show that if A is a subset of a universal set U, then

a) $A \oplus A = \emptyset$. **b)** $A \oplus \emptyset = A$.

c) $A \oplus U = \overline{A}$. **d)** $A \oplus \overline{A} = U$.

B.

Show that $A \oplus B = (A \cup B) - (A \cap B)$.

(a) Given: U is the universal set.

To proof: $A \oplus A = \emptyset$

PROOF

$$A \oplus A = \{x | x \in A \oplus A\}$$

By the definition of symmetric difference $A \oplus A$, x then has to be an element of A or an element of A, but not an element of both.

$$= \{x | (x \in A \vee x \in A) \wedge \neg(x \in A \wedge x \in A)\}$$

Use the idempotent laws for propositions:

$$= \{x | x \in A \wedge \neg(x \in A)\}$$

Use the negation law for propositions:

$$= \{x | \mathbf{F}\}$$

The empty set does not contain any elements and thus the statement $x \in \emptyset$ is always false.

$$= \{x | x \in \emptyset\}$$

$$= \emptyset$$

(b) Given: U is the universal set.

To proof: $A \oplus \emptyset = A$

PROOF

$$A \oplus \emptyset = \{x | x \in A \oplus \emptyset\}$$

By the definition of symmetric difference $A \oplus \emptyset$, x then has to be an element of A or an element of $\emptyset$, but not an element of both.

$$= \{x | (x \in A \vee x \in \emptyset) \wedge \neg(x \in A \wedge x \in \emptyset)\}$$

The empty set does not contain any elements and thus the statement $x \in \emptyset$ is always false.

$$= \{x | (x \in A \vee \mathbf{F}) \wedge \neg(x \in A \wedge \mathbf{F})\}$$

Use the identity law for propositions:

$$= \{x | x \in A \wedge \neg(x \in A \wedge \mathbf{F})\}$$

Use the domination law for propositions:

$$= \{x | x \in A \wedge \neg\mathbf{F}\}$$

$$= \{x | x \in A \wedge \mathbf{T}\}$$

Use the identity law for propositions:

$$= \{x | x \in A\}$$

$$= A$$

(c) Given: U is the universal set.

To proof: $A \oplus U = \overline{A}$

PROOF

$$A \oplus U = \{x | x \in A \oplus U\}$$

By the definition of symmetric difference $A \oplus U$, x then has to be an element of A or an element of U, but not an element of both.

$$= \{x | (x \in A \vee x \in U) \wedge \neg(x \in A \wedge x \in U)\}$$

The universal set contains all elements and thus the statement $x \in U$ is always true.

$$= \{x | (x \in A \vee \mathbf{T}) \wedge \neg(x \in A \wedge \mathbf{T})\}$$

Use the domination law for propositions:

$$= \{x | \mathbf{T} \wedge \neg(x \in A \wedge \mathbf{T})\}$$

Use the identity law for propositions:

$$= \{x | \mathbf{T} \wedge \neg(x \in A)\}$$

Use the identity law for propositions:

$$= \{x | \neg(x \in A)\}$$

Use the definition of the complement:

$$= \{x | x \in \overline{A}\}$$
$$= \overline{A}$$

(d) Given: U is the universal set.

To proof: $A \oplus \overline{A} = U$

PROOF

$$A \oplus \overline{A} = \{x | x \in A \oplus \overline{A}\}$$

By the definition of symmetric difference $A \oplus \overline{A}$, x then has to be an element of A or an element of $\overline{A}$, but not an element of both.

$$= \{x | (x \in A \lor x \in \overline{A}) \land \neg(x \in A \land x \in \overline{A})\}$$

Use the definition of the complement:

$$= \{x | (x \in A \lor \neg(x \in A)) \land \neg(x \in A \land \neg(x \in A))\}$$

Use the negation law for propositions:

$$= \{x | \mathbf{T} \land \neg(\mathbf{F})\}$$

$$= \{x | \mathbf{T} \land \mathbf{T}\}$$

Use the identity law for propositions:

$$= \{x | \mathbf{T}\}$$

The universal set contains all elements and thus the statement $x \in U$ is always true.

$$= \{x | x \in U\}$$

$$= U$$

To proof: $A \oplus B = (A \cup B) - (A \cap B)$

PROOF

$$A \oplus B = \{x | x \in A \oplus B\}$$

By the definition of symmetric difference $A \oplus B$, x then has to be an element of A or an element of B, but not an element of both.

$$= \{x | (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)\}$$

By the definition of the union:

$$= \{x | (x \in A \cup B) \wedge \neg(x \in A \wedge x \in B)\}$$

By the definition of the intersection:

$$= \{x | (x \in A \cup B) \wedge \neg(x \in A \cap B)\}$$

By the definition of the difference:

$$= \{x | x \in (A \cup B) - (A \cap B)\}$$

$$= (A \cup B) - (A \cap B)$$